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THE ROLLING OF A WHEEL WITH A PNEUMATIC TYRE ON A PLANE†

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A model of a pneumatic tyre as a system with an infinite number of degrees of freedom is proposed, when its surface is represented by the deformed surface of a torus. Using a number of hypotheses a functional of the potential energy of the deformations of the tyre is obtained as a function of the deformations of its tread. A complete system of equations of motion is obtained, assuming that the wheel rolls without slipping in the area of contact of the tread with the plane, with respect to the previously unknown part of the tread. In two special cases of the rolling of a wheel with breakaway and on a banking, all the characteristics of the motion (the contact area, the tyre deformation, and the forces and moments applied to the disc of the wheel) are obtained.

A number of models of a pneumatic tyre exist, which basically have a finite number of degrees of freedom and which are based on non-holonomic relations [1–3]. Dynamic effects, related to the deformation of the tyre over considerable parts of its free surface [4], can be described using a model of a tyre with an infinite number of degrees of freedom. Unlike the model proposed earlier [5], we consider the deformation of the whole surface of a torus, which models the tyre shape, in all directions and we determine the shape of the deformed pneumatic tyre both in the contact area and on its free surface. (In the previous model [5], the deformation is reduced to the displacement of the load line along the wheel axis, while the force and moment are proportional to this displacement and its derivative with respect to the natural parameter at the point of contact.)

1. A MODEL OF A WHEEL WITH A PNEUMATIC TYRE

We will assume that the wheel consists of a disc with an axis (1) (a solid), deformed by the side surface of the tyre (2) and an inextensible tread (3), along part of which contact occurs, without slipping, between the wheel and the plane OX_1X_2 (Fig. 1). The system of coordinates $Cx_1x_2x_3$ is obtained from the inertial system $OX_1X_2X_3$ by shifting the origin to the point C (the centre of mass of the undeformed wheel) and by rotation by an angle β around the CX_3 axis. The Cx_2 axis is the axis of rotation of the disc, while the plane Cx_1x_2 is the middle plane of the wheel and is orthogonal to the OX_1X_2 plane. Further, $\Gamma_2(\theta)$: $Cxyz \to Cx_1x_2x_3$ is the operator of rotation around the Cx_2 axis by an angle θ, while the system of coordinates $Cxyz$ is rigidly connected to the disc of the wheel (Fig. 2). We will assume that the side surface of the tyre in the undeformed state coincides with part of the surface of the torus. One can change to a toroidal system of coordinates $M\eta_1\eta_2\eta_3$ by means of the operator $\Gamma_2(\varphi)\Gamma_3(\psi)$ (Fig. 2). We will represent the radius vector of a point on the side surface of the tyre in the deformed state in the system of coordinates $OX_1X_2X_3$ in the form

$$\mathbf{R}(\varphi,\psi,t)=\sum_{i=1}^{3}X_i\mathbf{l}_i+\Gamma_3(\beta)\Gamma_2(\theta+\varphi)\left\{a\mathbf{e}_{x'}+\Gamma_3(\psi)\left[b\boldsymbol{\eta}_1+b\sum_{i=1}^{3}u_i(\varphi,\psi,t)\boldsymbol{\eta}_i\right]\right\},$$

$$\varphi \bmod 2\pi,\quad |\psi|\leq\psi_0 \tag{1.1}$$

$$\Gamma_2(\theta)=\begin{Vmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{Vmatrix},\quad \Gamma_3(\beta)=\begin{Vmatrix}\cos\beta & -\sin\beta & 0\\ \sin\beta & \cos\beta & 0\\ 0 & 0 & 1\end{Vmatrix}$$

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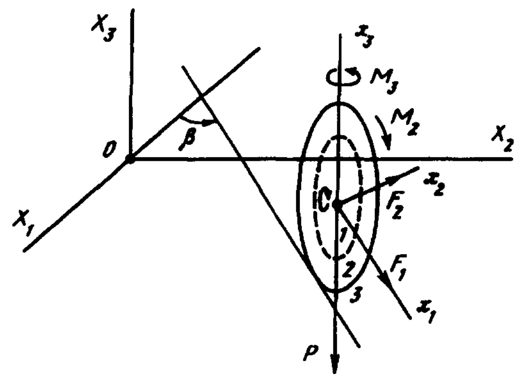

Fig. 1.

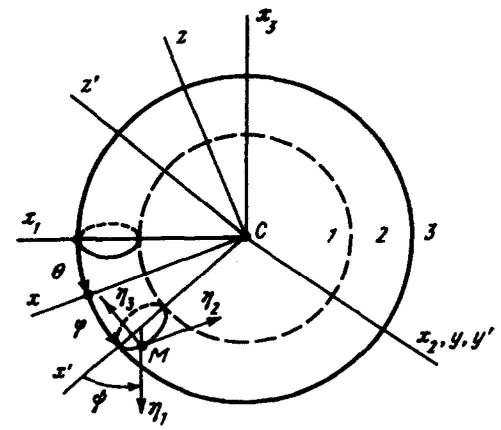

Fig. 2.

Here $\mathbf{l}_i$ is the unit vector if the OX_i axis, $\mathbf{e}_{x'}$ is the unit vector of the Cx' axis (Fig. 2), and $u_i(\varphi, \psi, t)$ is the projection of the displacement vector of a point on the tyre surface onto the axis with unit vector $\boldsymbol{\eta}_i$ of the toroidal system of coordinates.

We will formulate a number of hypotheses by means of which we can express the displacements of points on the tyre surface in terms of displacements of points of the tread. First, we will assume that the fibres of the tyre, corresponding to a constant value of the angle φ, are inextensible. Since the constant b is the radius of the circle obtained by a section of the torus with a plane passing through the Cx_2 axis, by (1.1) we obtain

$$\left(\frac{\partial \mathbf{R}}{\partial \psi}\right)^2 = b^2 \Rightarrow \left(\boldsymbol{\eta}_3 \times \boldsymbol{\eta}_1 + \boldsymbol{\eta}_3 \times \sum_{i=1}^{3} u_i \boldsymbol{\eta}_i + \sum_{i=1}^{3} \frac{\partial u_i}{\partial \psi} \boldsymbol{\eta}_i\right)^2 = 1$$

and further

$$(1 + u_1 + \partial u_2 / \partial \psi)^2 + (\partial u_1 / \partial \psi - u_2)^2 + (\partial u_3 / \partial \psi)^2 = 1 \qquad (1.2)$$

We will henceforth assume that the functions u_i and $\partial u_i/\partial \psi$ are small and we will represent (1.2), apart from second-order infinitesimals, in the form

$$u_1 + \partial u_2 / \partial \psi = 0 \qquad (1.3)$$

Second, we will assume that the tyre tread is also inextensible, i.e.

$$\left(\frac{\partial \mathbf{R}(\varphi, 0, t)}{\partial \varphi}\right)^2 = r^2 \Rightarrow \left(\boldsymbol{\eta}_2 \times \left[r\boldsymbol{\eta}_1 + b\sum_{i=1}^{3} u_i \boldsymbol{\eta}_i\right] + b\sum_{i=1}^{3} \frac{\partial u_i}{\partial \varphi} \boldsymbol{\eta}_i\right)^2 \Bigg|_{\psi=0} = r^2, \qquad r = a + b$$

and further

$$(1 + u + \partial v / \partial \varphi)^2 + (v - \partial u / \partial \varphi)^2 + (\partial w / \partial \varphi)^2 = 1 \qquad (1.4)$$

$$u(\varphi, t) = u_1(\varphi, 0, t) b / r, \quad v(\varphi, t) = -u_3(\varphi, 0, t) b / r$$

$$w(\varphi, t) = u_2(\varphi, 0, t) b / r$$

Note that the values of the angle $\psi = 0$ corresponds to points of the tread. Linearizing relation (1.4), we obtain

$$u + \partial v / \partial \varphi = 0 \qquad (1.5)$$

The functions u, v and w define the displacements of points of the tread in the toroidal system of coordinates, coinciding in this case, when $\psi = 0$, with the cylindrical coordinates $Cx'y'z'$, when $\mathbf{e}_{x'} = \boldsymbol{\eta}_1$, $\mathbf{e}_{y'} = \boldsymbol{\eta}_2$, $\mathbf{e}_{z'} = \boldsymbol{\eta}_3$.

Third, we will assume the curvature of the fibres of the tyre, corresponding to constant angle φ, to be constant. This hypothesis is based on the fact that an inextensible filament, clamped at the ends and acted upon by a constant distributed normal load (pressure), takes the form of a circle passing through the ends of the filament in the plane of action of the load. The fibre curvature k can be determined from the projections of the acceleration of a point on the axis of a Frenet trihedron, namely, $|\partial^2\mathbf{R}/\partial\psi^2| = b^2k$. From (1.1) we obtain

$$\left(\boldsymbol{\eta}_3\times\left[\boldsymbol{\eta}_3\times\left(\boldsymbol{\eta}_1+\sum_{i=1}^{3}u_i\boldsymbol{\eta}_i\right)\right]+\boldsymbol{\eta}_3\times\sum_{i=1}^{3}2\frac{\partial u_i}{\partial\psi}\boldsymbol{\eta}_i+\sum_{i=1}^{3}\frac{\partial^2 u_i}{\partial\psi^2}\boldsymbol{\eta}_i\right)^2=k^2b^2\Rightarrow$$

$$\Rightarrow\left(1+u_1+2\frac{\partial u_2}{\partial\psi}-\frac{\partial^2 u_1}{\partial\psi^2}\right)^2+\left(\frac{\partial^2 u_2}{\partial\psi^2}+2\frac{\partial u_1}{\partial\psi}-u_2\right)^2+\left(\frac{\partial^2 u_3}{\partial\psi^2}\right)^2=k^2b^2 \qquad (1.6)$$

Retaining terms of the zeroth and first order of smallness in (1.6), and taking (1.3) into account, we obtain

$$\partial^4u_2/\partial\psi^4+\partial^2u_2/\partial\psi^2=0 \qquad (1.7)$$

We will obtain a solution of Eq. (1.7) which, by (1.4) and (1.5), satisfies the conditions

$$u_2(\varphi,\pm\psi_0,t)=0,\quad u_2(\varphi,0,t)=w(\varphi,t)r/b$$
$$\partial u_2(\varphi,\pm\psi_0,t)/\partial\psi=0,\quad \partial u_2(\varphi,0,t)/\partial\psi=-u(\varphi,t)r/b \qquad (1.8)$$

The general solution of Eq. (1.7) has the form $u_2 = c_1\psi + c_2 + c_3\cos\psi + c_4\sin\psi$, where c_i are functions of φ and t. Taking boundary conditions (1.8) into account we obtain

$$u_2=u(\varphi,t)f(\psi,\psi_0)+w(\varphi,t)g(\psi,\psi_0)$$
$$u_1=-u(\varphi,t)f'(\psi,\psi_0)-w(\varphi,t)g'(\psi,\psi_0),\quad 0\leqslant\psi\leqslant\psi_0 \qquad (1.9)$$

$$f(\psi,\psi_0)=\frac{r}{b\Delta}[(1-\cos\psi_0)\psi+\Delta_1-\Delta_1\cos\psi+\Delta_2\sin\psi]$$

$$g(\psi,\psi_0)=\frac{r}{b\Delta}[-\sin\psi_0\psi-\Delta_2+(\cos\psi_0-1)\cos\psi+\sin\psi_0\sin\psi]$$

$$\Delta=\psi_0\sin\psi_0-2+2\cos\psi_0,\quad \Delta_1=\psi_0\cos\psi_0-\sin\psi_0,\quad \Delta_2=1-\cos\psi_0-\psi_0\sin\psi_0$$

In the range of values of ψ from $-\psi_0$ to zero, we need to replace ψ_0 by $-\psi_0$ in (1.9). Note that the function $f(\psi)$ is even while $g(\psi)$ is odd.

We expand the function $u_3(\varphi,\psi,t)$ in a Taylor series with respect to the variable ψ in the neighbourhood of the points $\psi = \psi_0$ and $\psi = -\psi_0$ and, confining ourselves in these expansions to the first two terms, we obtain

$$u_3(\varphi,\psi,t)=-\upsilon(\varphi,t)(1-|\psi|/\psi_0)r/b \qquad (1.10)$$

Hence, from the deformed state of the tread (the functions u, υ, w) we can determine the displacements of the points of the side surface of the tyre (formulae (1.9) and (1.10)). The shape of the deformed tyre is asymmetrical about the Cx_1x_3 plane, while the derivatives $\partial u_i(\varphi,\psi,t)/\partial\psi$ may have discontinuities when $\psi = 0$.

We will calculate the elementary work of the pressure in the tyre for possible displacements of points on its surface. We have

$$\delta A=\int_{-\psi_0}^{\psi_0}\int_0^{2\pi}p\mathbf{n}\delta\mathbf{R}d\sigma,\quad \delta\mathbf{R}=b\sum_{i=1}^{3}\delta u_i\boldsymbol{\eta}_i$$

$$\mathbf{n}d\sigma=\left[\frac{\partial\mathbf{R}}{\partial\varphi}\times\frac{\partial\mathbf{R}}{\partial\psi}\right]d\psi d\varphi$$

and further, apart from terms of the second order of smallness inclusive

$$\delta A = pb^3 \int_{-\psi_0}^{\psi_0}\int_0^{2\pi}\left[\delta u_1\left(\frac{a}{b}+\cos\psi-\frac{\partial u_3}{\partial\varphi}+u_1\cos\psi-u_2\sin\psi\right)+\right.$$
$$\left.+\delta u_2\left(\frac{a}{b}+\cos\psi\right)\left(u_2-\frac{\partial u_1}{\partial\psi}\right)+\delta u_3\left(\frac{\partial u_1}{\partial\varphi}+u_3\cos\psi\right)\right]d\psi d\varphi \tag{1.11}$$

In relations (1.11) we must replace u_i and δu_i by expressions (1.9) and (1.10) and integrate them over ψ.

The integrand in (1.11) contains the term $\delta u_1(a/b + \cos\psi)$, linear in δu_1. To retain the assumed approximation accuracy we must obtain u_1 to second-order infinitesimals inclusive, with respect to u, υ and w. From relations (1.2) and (1.6), assuming $u_i = u_{i0} + z_i$, where u_{i0} are the functions (1.9) and (1.10) and z_i are corrections, quadratic in u, υ and w, we obtain, to second-order infinitesimals inclusive (the prime denotes a derivative with respect to ψ)

$$z_1''+z_1=\frac{1}{2}u_{30}''^2-u_{30}'^2-\frac{1}{2}(u_{10}'-u_{20})^2 \tag{1.12}$$

The right-hand side of Eq. (1.12) is equal to $-1/2[u(f''+f)+w(g''+g)]^2-\upsilon^2r^2(b\psi_0)^{-2}$, while the function z_i vanishes when $\psi = 0$ and $\psi = \pm\psi_0$. The solution of Eq. (1.12) has the form

$$z_1(\varphi,\psi,t)=A_{11}(\psi)u^2+A_{33}(\psi)w^2+2A_{13}(\psi)uw+A_{22}(\psi)\,\upsilon^2$$

where A_{ii} $(i = 1, 2, 3)$ is an even function while A_{13} is an odd function. The functions A_{ij} are specified in the range $[-\psi_0, \psi_0]$ and are linear combinations of ψ, ψ^2, $\cos\psi$, $\sin\psi$ and are constant. No difficulties arise in determining these functions although it is a fairly long process. Substituting (1.9) and (1.10) into (1.11), replacing δu_1 by $-f'\delta u - g'\delta w + \delta z_1$ and integrating the expressions obtained, taking into account the evenness and oddness of the corresponding functions with respect to ψ, we obtain an expression for the work done by the pressure in possible displacements in the form

$$\delta A=-\int_0^{2\pi}[n_0\delta u+n_1u\delta u+n_2\upsilon\delta\upsilon+n_3w\delta w+n_{12}(u'\delta\upsilon-\upsilon'\delta u)]d\varphi \tag{1.13}$$

It is a fairly lengthy process to calculate the coefficients n_k $(k = 0, \ldots, 3)$ and n_{12} in explicit form.

Note that $n_0 < 0$, while the variation (1.13), taking (1.5) into account, can be represented in the form

$$\delta A=-\delta\Pi[\upsilon,w],\quad \Pi=\frac{1}{2}\int_0^{2\pi}[(n_1+2n_{12})\upsilon'^2+n_2\upsilon^2+n_3w^2]d\varphi$$

where Π is the potential energy of the deformed tyre. When $u = \upsilon = w = 0$ the tyre is in stable equilibrium, and this means that the functional Π has an isolated minimum and the coefficients $n_1 + 2n_{12}$, n_2, n_3 are positive.

The second note touches on the constancy of the pressure p. If we assume that the gas in the pneumatic tyre is a perfect gas, and the processes are isothermal, we have $pV = p_0V_0$, where p, V and p_0, V_0 are the pressure and volume of the gas in the deformed and undeformed tyre, respectively. Then

$$p=p_0\left(1+\frac{\Delta V}{V_0}\right)^{-1}=p_0\left(1-\frac{\Delta V}{V_0}+\ldots\right),\quad V=V_0+\Delta V$$
$$\Delta V=b^3\int_{-\psi_0}^{\psi_0}\int_0^{2\pi}u_1\left(\frac{a}{b}+\cos\psi\right)d\psi d\varphi+O_2 \tag{1.14}$$

where O_2 are second-order and higher infinitesimals in u, υ, w and their derivatives. Taking (1.5) and (1.9) and the oddness of the function $g'(\psi)$ into account, we arrive at the conclusion that the quantity ΔV in (1.14) is of the second and higher order of smallness in u, υ and w and, consequently, the pressure p in (1.11) can be assumed constant, which corresponds to the assumed accuracy when calculating the work done by the pressure in possible displacements.

2. THE EQUATIONS OF MOTION OF THE WHEEL WITH THE TYRE

The kinetic energy of the wheel is made up of the kinetic energy of the disc

$$T_d = \frac{1}{2} m_d \sum_{i=1}^{3} \dot{X}_i^2 + \frac{1}{2} J_{1d}\dot{\beta}^2 + \frac{1}{2} J_{2d}\dot{\theta}^2$$

where m_d is the mass of the disc and J_{1d}, J_{2d} are the moments of inertia of the disc about the axes Cx_3 and Cx_2, and the kinetic energy of the deformed tyre. As regards the latter we can assume that the whole mass of the tyre is concentrated in the tread (a uniform inextensible filament), and we can represent the kinetic energy of the tread in the form

$$T_b = \frac{1}{2}\rho r \int_0^{2\pi} \dot{\mathbf{R}}^2(\varphi,0,t)d\varphi, \quad \Phi = \theta + \varphi$$

$$\dot{\mathbf{R}}^2(\varphi,0,t) = \sum_{i=1}^{3} \dot{X}_i \mathbf{l}_i + r\Gamma_3(\beta)\{\dot{\beta}\mathbf{l}_3 \times \Gamma_2(\Phi)[(1+u)\boldsymbol{\eta}_1 + w\boldsymbol{\eta}_2 - v\boldsymbol{\eta}_3]\} +$$
$$+ r\Gamma_3(\beta)\Gamma_2(\Phi)\{\dot{\theta}\boldsymbol{\eta}_2 \times [(1+u)\boldsymbol{\eta}_1 + w\boldsymbol{\eta}_2 - v\boldsymbol{\eta}_3] + \dot{u}\boldsymbol{\eta}_1 + \dot{w}\boldsymbol{\eta}_2 - \dot{v}\boldsymbol{\eta}_3\}$$

where ρ is the density per unit length of the tread. Further, we obtain

$$\Gamma_2(-\Phi)\Gamma_3(-\beta)\dot{\mathbf{R}}(\varphi,0,t) = \sum_{i=1}^{3} Z_i \boldsymbol{\eta}_i, \quad Z_i = \xi_{i1} + r\zeta_{i2} \tag{2.1}$$

$$\zeta_{11} = \dot{X}_1 \cos\Phi\cos\beta + \dot{X}_2 \cos\Phi\sin\beta - \dot{X}_3 \sin\Phi, \quad \zeta_{12} = \dot{u} - v\dot{\theta} - \dot{\beta}w\cos\Phi$$
$$\zeta_{21} = -\dot{X}_1 \sin\beta + \dot{X}_2 \cos\beta, \quad \zeta_{22} = \dot{w} + \dot{\beta}(1+u)\cos\Phi - \dot{\beta}v \sin\Phi$$
$$\zeta_{31} = \dot{X}_1 \sin\Phi\cos\beta + \dot{X}_2 \sin\Phi\sin\beta + \dot{X}_3 \cos\Phi, \quad \zeta_{32} = -\dot{v} - \dot{\theta}(1+u) - \dot{\beta}w\sin\Phi$$

and hence the kinetic energy of the wheel is

$$T = T_d + T_b = \frac{m}{2}\sum_{i=1}^{3} \dot{X}_i^2 + \frac{1}{2}J_1\dot{\beta}^2 + \frac{1}{2}J_2\dot{\theta}^2 +$$
$$+ \frac{\rho r}{2}\int_0^{2\pi} \{r^2[\xi_{12}^2 + (\zeta_{22} - \dot{\beta}\cos\Phi)^2 + (\zeta_{32} + \dot{\theta})^2\} +$$
$$+ 2r[\zeta_{11}\zeta_{12} + (\zeta_{21} + r\dot{\beta}\cos\Phi)(\zeta_{22} - \dot{\beta}\cos\Phi) + (\zeta_{31} - r\dot{\theta})(\zeta_{32} + \dot{\theta})]\}d\varphi \tag{2.2}$$

Here m, J_1 and J_2 are the mass of the wheel and its moments of inertia in the undeformed state about the axes Cx_3 and Cx_2.

We will assume that the wheel rolls on the OX_1X_2 plane without slipping. This means that in the range $[\varphi_1, \varphi_2]$ of variation of the angle φ, the velocity of points on the tread is equal to zero. From (2.1) we obtain

$$Z_i = 0, \quad i = 1,2,3, \quad \varphi \in [\varphi_1, \varphi_2] \tag{2.3}$$

and the possible displacements satisfy the conditions

$$\delta Z_i = 0, \quad i = 1,2,3, \quad \varphi \in [\varphi_1, \varphi_2] \tag{2.4}$$

Relations (2.3) can be replaced by the single holonomic relation $\mathbf{R}(\varphi, 0, t)\mathbf{l}_3 = 0$ and two non-holonomic relations, for example, $Z_2 = 0$ and $Z_3 = 0$. Moreover, at the boundary points of contact between the tread and the plane, corresponding to the angle φ_1 and φ_2, we will introduce two constraint actions $\nu_1(t)$ and $\nu_2(t)$, which satisfy the conditions

$$\Gamma_2(-\Phi)\Gamma_3(-\beta)\mathbf{l}_3\boldsymbol{\nu}_k = 0 \Rightarrow \nu_{1k}\sin\Phi_k - \nu_{3k}\cos\Phi_k = 0 \tag{2.5}$$
$$\boldsymbol{\nu}_k = (\nu_{1k}, \nu_{2k}, \nu_{3k}), \quad \Phi_k = \varphi_k + \theta; \quad k = 1,2$$

Conditions (2.5) denote that the constraint actions at the boundary points of the contact line are equal to zero in the projection onto the OX_3 axis. The work of these forces in possible displacements, after eliminating the constraints, is

$$\delta A_k = \sum_{i=1}^{3} \nu_{ik}\delta Z_{ik}, \quad k=1,2; \qquad \delta Z_{ik} = \delta Z_i|_{\varphi=\varphi_k} \tag{2.6}$$

When eliminating constraints (2.3) one must also take into account the work done by the constraint actions $\boldsymbol{\mu}(\varphi, t)$, $\varphi_1 \leqslant \varphi \leqslant \varphi_2$, defined in the form

$$\delta A_\mu = \int_{\varphi_1}^{\varphi_2} \sum_{i=1}^{3} \mu_i(\varphi,t)\delta Z_i d\varphi \tag{2.7}$$

We will assume that a force and a moment are applied to the wheel disc (Fig. 1), the work of which in possible displacements is

$$\delta A_F = F(\beta)\delta X_1 + F(\beta-\pi/2)\delta X_2 - P\delta X_3 + M_2\delta\theta + M_3\delta\beta \tag{2.8}$$
$$F(\beta) = F_1\cos\beta - F_2\sin\beta$$

The equations of motion of the wheel and the conditions where the functions undergo a jump at the boundary points of the contact line are obtained from Hamilton's variational principle

$$\int_{t_1}^{t_2} (\delta T + \delta A + \delta A_1 + \delta A_2 + \delta A_F + \delta A_\mu + \delta A_3)dt = 0 \tag{2.9}$$

$$\delta A_3 = \int_{\varphi_2}^{2\pi-\varphi_1} \lambda(\varphi,t)[(1+u+v')(\delta u + \delta v') + (v-u')(\delta v - \delta u') + w'\delta w']d\varphi$$

where $\lambda(\varphi, t)$ is a Lagrange multiplier, corresponding to the condition for the tread to be inextensible (1.4), while the remaining quantities are given by (1.13) and (2.6)–(2.8). The integration domain $[t_1, t_2] \cup [0, 2\pi]$ in (2.9) is split by the curves $\varphi = \varphi_1(t)$ and $\varphi = \varphi_2(t)$ into two parts, to each of which Green's formula is applied. Hence we obtain the following system of equations

$$-\frac{d}{dt}\nabla_{\dot{X}_1}T + \int_{\varphi_1}^{\varphi_2} S_1(\mu_i,\Phi,\beta)d\varphi + \sum_{k=1}^{2} S_1(\nu_{ik},\Phi_k,\beta) + F(\beta) = 0$$

$$-\frac{d}{dt}\nabla_{\dot{X}_2}T + \int_{\varphi_1}^{\varphi_2} S_1\left(\mu_i\,\Phi,\beta-\frac{\pi}{2}\right)d\varphi + \sum_{k=1}^{2} S_1\left(\nu_{ik},\Phi_k,\beta-\frac{\pi}{2}\right) + F\left(\beta-\frac{\pi}{2}\right) = 0$$

$$S_1(\mu_i,\Phi,\beta) = \mu_1\cos\Phi\cos\beta - \mu_2\sin\beta + \mu_3\sin\Phi\cos\beta$$

$$-\frac{d}{dt}\nabla_{\dot{X}_3}T - \int_{\varphi_1}^{\varphi_2} S_2(\mu_i,\Phi)d\varphi - \sum_{k=1}^{2} S_2(\nu_{ik},\Phi_k) - P = 0 \tag{2.10}$$

$$S_2(\mu_i,\Phi) = \mu_1\sin\Phi - \mu_3\cos\Phi$$

$$\nabla_\theta T - \frac{d}{dt}\nabla_{\dot{\theta}}T - r\int_{\varphi_1}^{\varphi_2} S_3(\mu_i,u,v)d\varphi - r\sum_{k=1}^{2} S_3(\nu_{ik},u_k,v_k) + M_2 = 0$$

$$S_3(\mu_i,u,v) = \mu_1 v + \mu_3(1+u)$$

$$\nabla_\beta T - \frac{d}{dt}\nabla_{\dot{\beta}}T - r\int_{\varphi_1}^{\varphi_2} S_4(\mu_i,\Phi,u,v,w)d\varphi - r\sum_{k=1}^{2} S_4(\nu_{ik},\Phi_k,u_k,v_k,w_k) + M_3 = 0$$

$$S_4(\mu_i,\Phi,u,v,w) = \mu_1 w\cos\Phi - \mu_2((1+u)\cos\Phi - v\sin\Phi) + \mu_3 w\sin\Phi$$

$$u_k = u(\varphi_k,t),\quad v_k = v(\varphi_k,t),\quad w_k = w(\varphi_k,t),\quad k=1,2$$

$$\nabla_u T - \frac{d}{dt}\nabla_{\dot{u}}T - n_0 - n_1 u + n_{12}v' + \lambda(1+u+v') + [\lambda(v-u')]' = 0,\quad \varphi\in I_2$$

$$-n_0 - n_1 u + n_{12}v' + \mu_1 r = 0,\quad \varphi\in I_1$$

$$\rho r^3\dot\varphi_k[\dot u]_k-(-1)^{k+1}\lambda(\upsilon-u')|_{l(k)}+r\nu_{1k}=0$$
$$\nabla_\upsilon T-\frac{d}{dt}\nabla_{\dot\upsilon}T-n_2\upsilon-n_{12}u'+\lambda(\upsilon-u')-[\lambda(1+u+\upsilon')]'=0,\quad \varphi\in I_2$$
$$n_2\upsilon+n_{12}u'+\mu_3 r=0,\quad \varphi\in I_1$$
$$\rho r^3\dot\varphi_k[\dot\upsilon]_k+(-1)^{k+1}\lambda(1+u+\upsilon')|_{l(k)}-\nu_{3k}r=0$$
$$\nabla_w T-\frac{d}{dt}\nabla_{\dot w}T-n_3w-(\lambda w')'=0,\quad \varphi\in I_2$$
$$n_3w=\mu_2 r,\quad \varphi\in I_1$$
$$\rho r^3\dot\varphi_k[\dot w]_k+(-1)^{k+1}\lambda w'|_{l(k)}+r\nu_{2k}=0$$
$$k=1,2;\quad I_1=]\varphi_1,\varphi_2[,\quad I_2=]\varphi_2,2\pi-\varphi_1[$$

Here $[f(\varphi,t)]_k=f(\varphi_k+0,t)-f(\varphi_k-0,t)$ is the jump of the function at the point φ_k, while the subscripts $l(1)$ and $l(2)$ denote the limits of the corresponding functions as $\varphi\to\varphi_1$ from the left and $\varphi\to\varphi_2$ from the right. Relations (2.10), in addition to the equations of motion, contain the joining conditions (the conditions at the jump) on the boundaries of the contact area when $\varphi\to\varphi_1$ and $\varphi\to\varphi_2$; together with the constraint equations (1.4) and (2.3) and conditions (2.5) they form a complete set of equations of the problem (20 relations in all) for the 20 unknowns: X_i, μ_i, ν_{i1}, ν_{i2} $(i=1,2,3)$, β, θ, u, υ, w, λ, φ_1, φ_2. In addition, when determining the functions u, υ and w one must take into account their continuity at the points φ_1 and φ_2, namely, $[u]_k=[\upsilon]_k=[w]_k=0$ $(k=1,2)$.

3. ROLLING OF THE WHEEL WITH BREAKAWAY

We will consider special cases of the above problem, namely, the rolling of a wheel with breakaway and motion on a banking. In these cases we can obtain an analytic solution of the problem and determine the forces and moments necessary for these situations to occur.

Consider the rolling of a wheel with breakaway, when

$$\beta=\dot\beta=0,\quad \dot X_1=c\cos\varepsilon,\quad \dot X_2=c\sin\varepsilon,\quad X_3=\text{const},\quad \dot\theta=\Omega$$
$$(u,\upsilon,w)(\varphi,t)=(U,V,W)(\alpha),\quad \alpha=\varphi+\Omega t-\pi/2,\quad \mu(\varphi,t)=\mu(\alpha)$$
$$\dot\varphi_k=-\Omega,\quad \nu_k=\text{const},\quad k=1,2,\quad \lambda(\varphi,t)=\lambda(\alpha)$$

where ε is the constant angle of breakaway. The equations of motion (2.10) for the functions u, υ, and w in the contact area and the conditions for rolling without slipping (2.3) can be represented in the form (the primes denote a derivative with respect to α)

$$\begin{gathered}\mu_1 r=n_0+n_1U-n_{12}V',\quad \mu_2 r=n_3W,\quad -\mu_3 r=n_2V+n_{12}U'\\ \alpha\in]\alpha_1,\alpha_2[,\quad \alpha_k=\varphi_k+\Omega t-\pi/2\\ c\cos\varepsilon\sin\alpha=r\Omega(U'-V),\quad c\sin\varepsilon=-r\Omega W'\\ c\cos\varepsilon\cos\alpha=r\Omega(1+U+V')\end{gathered}\tag{3.1}$$

Assuming that the wheel centre moves along a straight line $L=\{X_2=X_1\operatorname{tg}\varepsilon, X_3=\text{const}\}$ with constant velocity c, we will seek a solution of the last three equations of system (3.1), which define the contact area, taking into account the inextensibility of the tread (1.4) in the form of a section of a straight line parallel to L. As a result we obtain

$$\begin{gathered}U=d\alpha\sin\alpha+d_1\sin\alpha+r^{-1}X_3\cos\alpha-1\\ V=d\alpha\cos\alpha+d_1\cos\alpha-r^{-1}X_3\sin\alpha\\ W=-d\operatorname{tg}\varepsilon\alpha+d_3,\quad d=c\cos\varepsilon(r\Omega)^{-1}\end{gathered}\tag{3.2}$$

where d_1 and d_3 are arbitrary constants. The equations of the contact line in the system of coordinates $OX_1X_2X_3$ have the form

$$\begin{gathered}\xi_1=c\cos\varepsilon t-c\cos\varepsilon\Omega^{-1}\alpha-rd_1\\ \xi_2=c\sin\varepsilon t-c\sin\varepsilon\Omega^{-1}\alpha+rd_3,\quad \xi_3=0;\quad \alpha\in[\alpha_1,\alpha_2]\end{gathered}\tag{3.3}$$

It follows from the condition for the tread to be inextensible, that when the angle α changes, the arc of the tread $r\alpha$ coincides with the corresponding section of the contact area, which is possible when $c = r\Omega$. The contact line, by (3.3), is a straight line parallel to the straight line L. Further, the reactions in the contact area μ_1, μ_2, μ_3 are found from the first three equations of (3.1).

The shape of the deformed tread outside the contact area can be found by solving Eqs (2.10) for the functions u, υ and w, represented in the form

$$\begin{aligned}
&g_0(1+U-U''+2V')-n_0-n_1U+n_{12}V'+\lambda(1+U+V')+\lambda(V'-U'')+\lambda'(V-U')=0\\
&g_0(V-V''-2U')-n_2V-n_{12}U'-\lambda'(1+U+V')-\lambda(U'+V'')+\lambda(V-U')=0\\
&g_0W''+n_3W+\lambda'W'+\lambda W''=0,\quad U+V'=0;\quad g_0=\rho r^3\Omega^2;\quad \alpha\in[\alpha_2,2\pi-\alpha_1]
\end{aligned} \tag{3.4}$$

The last equation in (3.4) is the linearized condition for the tread to be inextensible (1.5). Assuming that the tension in the tread is $-\lambda = g_0 - n_0 - \nu(\alpha)$, where $\nu(\alpha)$ is a small quantity, and only linear terms remain in (3.4), we obtain the general solution of the corresponding linear system in the form

$$V=\sum_{k=1}^{4} C_k\exp(D_k\alpha),\quad W=A_1\exp(\gamma\alpha),\ +A_2\exp(-\gamma\alpha),\quad U=-V';\quad \alpha\in[\alpha_2,2\pi-\alpha_1] \tag{3.5}$$

$$D_k=\pm\left[\tilde n\pm\left(\tilde n^2-1+\frac{n_2}{n_0}\right)^{1/2}\right]^{1/2},\quad \tilde n=1+\frac{n_1/2+n_{12}}{n_0},\quad \gamma=\left(-\frac{n_3}{n_0}\right)^{1/2}>0$$

The form of the roots D_k depends on the geometrical characteristics of the tyre (the quantities a, b and ψ_0). The coefficients C_k and A_k are found from the conditions imposed on the jump in (2.10), and the conditions for the functions U, V and W to be continuous at the boundary points of the contact area K_1 and K_2, namely

$$\begin{aligned}
&g_0[U']_k+(-1)^{k+1}(n_0-g_0)(V-U')_{l(k)}=r\nu_{1k},\quad [U]_k=0\\
&g_0[V']_k-(-1)^{k+1}(n_0-g_0+\nu)_{l(k)}=-r\nu_{3k},\quad [V]_k=0\\
&g_0[W']_k-(-1)^{k+1}(n_0-g_0)W'_{l(k)}=r\nu_{2k},\quad [W]_k=0;\quad k=1,2
\end{aligned} \tag{3.6}$$

Using (3.2) and (3.5) we will represent the joining conditions (3.6) in the form

$$\sum_{k=1}^{4} D_k^m W_{kj}=a_{mj},\quad m=0,1,2,3 \tag{3.7}$$

$$\begin{aligned}
&a_{0j}=(\cos\varepsilon-1)\alpha_j+d_1,\quad a_{1j}=d_2=1-X_3r^{-1}\\
&a_{2j}=(1-\cos\varepsilon)\alpha_j-d_1-\frac{g_0}{n_0}\cos\varepsilon\alpha_j-(-1)^j\frac{r\nu_{1j}}{n_0}\\
&a_{3j}=1-\left(1+\frac{n_1+n_{12}}{n_0}\right)d_2-\frac{g_0}{n_0}\cos\varepsilon+(-1)^j\frac{r\nu_{3j}}{n_0}\\
&W_{k1}=C_k\exp[D_k(2\pi-\alpha_1)]\approx C_k\exp(2\pi D_k)\\
&W_{k2}=C_k\exp(D_k\alpha_2)\approx C_k\\
&A_m\exp\{(-1)^{m+1}\gamma[2\pi(2-s)+(-1)^s\alpha_s]\}=G_{ms}\\
&G_{ms}=\frac{1}{2\gamma}\left[\gamma d_3+(-1)^m\frac{g_0\sin\varepsilon}{n_0}-\alpha_s\gamma\sin\varepsilon-(-1)^{m+s}\frac{r\nu_{2s}}{n_0}\right];\quad m,s=1,2
\end{aligned}$$

The quantities W_{kj} are found from the system of linear algebraic equations (3.7) in the form

$$W_{kj}=\left[\prod_{i\neq k}^{4}(D_i-D_k)\right]^{-1}[a_{0j}D_k^{-1}D_1^2D_3^2+a_{1j}(D_1^2+D_3^2-D_k^2)-a_{2j}D_k-a_{3j}]$$

$$k=1,\dots,4,\quad j=1,2 \tag{3.8}$$

Assuming $\exp(D_k\alpha_j) \approx 1$ and $\exp(\pm\gamma\alpha_j) \approx 1$, from (3.7) and (3.8) we obtain the relations

$$W_{k1}\exp(-\pi D_k) = W_{k2}\exp(\pi D_k), \quad k = 1,\ldots,4$$
$$G_{j1}\exp(-\gamma\pi) = G_{j2}\exp(\gamma\pi), \quad j = 1,2$$

from which we obtain the equations

$$\begin{aligned}
&\frac{r}{n_0}(\nu_{11}+\nu_{12}) = \left[(1-\cos\varepsilon)(1+H_1) - \frac{g_0}{n_0}\cos\varepsilon\right](\alpha_2-\alpha_1) + 2H_2 d_2\\
&\frac{r}{n_0}(\nu_{11}-\nu_{12}) = [2d_1 - (1-\cos\varepsilon)(\alpha_1+\alpha_2)](1+H_3)\\
&\frac{r}{n_0}(\nu_{31}+\nu_{32}) = -[2d_1 - (1-\cos\varepsilon)(\alpha_1+\alpha_2)]H_4\\
&\frac{r}{n_0}(\nu_{31}-\nu_{32}) = 2 - \frac{2g_0\cos\varepsilon}{n_0} - 2\left(1+\frac{n_1+n_{12}}{n_0}+H_5\right)d_2 - (1-\cos\varepsilon)H_6(\alpha_2-\alpha_1)\\
&\frac{r}{n_0}(\nu_{21}+\nu_{22}) = \gamma\,\mathrm{th}\,\pi\gamma[\sin\varepsilon(\alpha_1+\alpha_2) - 2d_3]\\
&\frac{r}{n_0}(\nu_{21}-\nu_{22}) = -\frac{2g_0\sin\varepsilon}{n_0} - \gamma\sin\varepsilon\,\mathrm{cth}\,\pi\gamma(\alpha_2-\alpha_1)\\
&H_1 = D_1D_3G_{13}/G_{31}, \quad H_2 = (D_3^2-D_1^2)\,\mathrm{th}\,\pi D_1\,\mathrm{th}\,\pi D_3/G_{31}\\
&H_3 = D_1D_3G_{31}/G_{13}, \quad H_4 = D_1D_3(D_3^2-D_1^2)\,\mathrm{th}\,\pi D_1\,\mathrm{th}\,\pi D_3/G_{13}\\
&H_5 = (D_3^3\,\mathrm{th}\,\pi D_1 - D_1^3\,\mathrm{th}\,\pi D_3)/G_{31}, \quad H_6 = D_1D_3(D_1^2-D_3^2)/G_{31}\\
&G_{ik} = D_i\,\mathrm{th}\,\pi D_1 - D_k\,\mathrm{th}\,\pi D_3, \quad i,k = 1,3
\end{aligned} \tag{3.9}$$

In relations (3.8) and (3.9) we have stipulated that $D_2 = -D_1$, and $D_4 = -D_3$. The terms in first-order infinitesimals remaining in the first five equations of (2.10) and in conditions (2.5) can be represented in the form

$$\begin{aligned}
&F_1 = -\nu_{31} - \nu_{32} = r^{-1}n_0H_4[2d_1 - (1-\cos\varepsilon)(\alpha_1+\alpha_2)], \quad M_2 = -rF_1\\
&F_2 = -\nu_{21} - \nu_{22} = r^{-1}n_0\gamma\,\mathrm{th}\,\pi\gamma[2d_3 - \sin\varepsilon(\alpha_1+\alpha_2)]\\
&P = -r^{-1}n_0(\alpha_2-\alpha_1), \quad M_3 = n_0\sin\varepsilon(\alpha_2-\alpha_1)\\
&\nu_{11}+\nu_{12}+\nu_{31}\alpha_1+\nu_{32}\alpha_2 = 0 \Rightarrow 2H_2d_2 + [H_1 - (1+H_1)\cos\varepsilon](\alpha_2 -\\
&\nu_{11}-\nu_{12}+\nu_{31}\alpha_1-\nu_{32}\alpha_2 = 0 \Rightarrow \left(1-\frac{g_0\cos\varepsilon}{n_0}\right)(\alpha_1+\alpha_2)+\\
&+[2d_1 - (1-\cos\varepsilon)(\alpha_1+\alpha_2)](1+H_3) = 0
\end{aligned} \tag{3.10}$$

It follows from relations (3.10) that the conditions for steady rolling of the wheel with breakaway to exist are the equations $M_2 = -rF_1$, $M_3 = -Pr\sin\varepsilon$. The remaining characteristics of the steady state are found from (3.10) in the form

$$\begin{aligned}
&\alpha_2-\alpha_1 = -n_0^{-1}Pr, \quad \alpha_1+\alpha_2 = -\left(1-\frac{g_0\cos\varepsilon}{n_0}\right)^{-1}(1+H_3)H_4^{-1}n_0^{-1}F_1r\\
&2d_1 = H_4^{-1}\left[1-(1-\cos\varepsilon)\left(1-\frac{g_0\cos\varepsilon}{n_0}\right)^{-1}(1+H_3)\right]n_0^{-1}F_1r\\
&2d_3 = \mathrm{cth}\,\pi\gamma n_0^{-1}F_2r - \sin\varepsilon\left(1-\frac{g_0\cos\varepsilon}{n_0}\right)^{-1}(1+H_3)H_4^{-1}n_0^{-1}F_1r
\end{aligned} \tag{3.11}$$

Note that there is no resistance to the rolling of the wheel with breakaway. The latter manifests itself if dissipative forces related to the deformation of the tyre are introduced.

4. THE ROLLING OF A WHEEL ON A BANKING

We will consider one other steady state of the rolling of the wheel on a banking, when

$$\begin{gathered} X_1 = R\sin\omega t,\quad X_2 = -R\cos\omega t,\quad X_3 = \text{const} \\ \beta = \omega t,\quad \omega = \text{const},\quad \dot\theta = \Omega,\quad \dot\varphi_k = -\Omega = \text{const},\quad k = 1,2 \\ (u, v, w)(\varphi, t) = (U, V, W)(\alpha),\quad \alpha = \varphi + \Omega t - \pi/2 \\ \mu(\varphi, t) = \mu(\alpha),\quad \nu_k = \text{const},\quad k = 1,2,\quad \lambda(\varphi, t) = \lambda(\alpha) \end{gathered} \tag{4.1}$$

Here R and $R\omega$ is the radius of the banking and the velocity of the wheel centre. The first three equations of (3.1), in the case when the wheel rolls on a banking, retain their form, while the last three equations are written in the form

$$\begin{gathered} \Omega U' - \Omega V + \omega W\sin\alpha = r^{-1}R\omega\sin\alpha \\ \Omega W' - \omega(1+U)\sin\alpha - \omega V\cos\alpha = 0 \\ \Omega V' + \Omega(1+U) + \omega W\cos\alpha = r^{-1}R\omega\cos\alpha \end{gathered} \tag{4.2}$$

Equations (4.2) allow of a first integral

$$(1+U)\cos\alpha - V\sin\alpha = X_3 r^{-1} \tag{4.3}$$

while the condition for the tread to be inextensible (1.4) becomes

$$W'^2 + \left(\frac{\omega}{\Omega}W - \frac{R\omega}{r\Omega}\right)^2 = 1 \tag{4.4}$$

The general solution of Eq. (4.4) has the form

$$W = \frac{R}{r} - \frac{\Omega}{\omega}\cos\left(\frac{\omega}{\Omega}\alpha + \delta\right),\quad \alpha \in [\alpha_1, \alpha_2] \tag{4.5}$$

where δ is an arbitrary constant. Further, we obtain from the second equation of (4.2) and (4.3)

$$\begin{gathered} U = \frac{\Omega}{\omega}\sin\left(\frac{\omega}{\Omega}\alpha + \delta\right)\sin\alpha + \frac{X_3}{r}\cos\alpha - 1 \\ V = \frac{\Omega}{\omega}\sin\left(\frac{\omega}{\Omega}\alpha + \delta\right)\cos\alpha - \frac{X_3}{r}\sin\alpha,\quad \alpha \in [\alpha_1, \alpha_2] \end{gathered} \tag{4.6}$$

We obtain the equations describing the contact area in the system of coordinates $OX_1X_2X_3$ from (1.1) with $\psi = 0$ in the form

$$\xi_1 = r\frac{\Omega}{\omega}\sin\left(\beta - \frac{\omega}{\Omega}\alpha - \delta\right),\quad \xi_2 = -r\frac{\Omega}{\omega}\cos\left(\beta - \frac{\omega}{\Omega}\alpha - \delta\right),\quad \xi_3 = 0 \tag{4.7}$$

It follows from relations (4.7) that the contact area is an arc of a circle with centre at the point O and radius $r\Omega/\omega$. Since the quantities U, V, W, α_1, α_2 are small, we arrive at the conclusion that the quantities $\delta_1 = \Omega/\omega - R/r$ and δ are also small. Hence, in the contact area we obtain, apart from second-order infinitesimals

$$\begin{gathered} U = -d_2,\quad V = r^{-1}R\delta,\quad W = -\delta_1,\quad d_2 = 1 - r^{-1}X_3 \\ U' = \alpha + \delta R/r,\quad V' = d_2,\quad W' = \delta + r\alpha/R \end{gathered}$$

The shape of the deformed tread and its tension are found from Eqs (3.4)–(3.6). Equations (3.7) have the same form, only their right-hand sides must be taken as follows:

$$a_{0j}=\delta R/r,\quad a_{1j}=d_2$$

$$a_{2j}=-\frac{R}{r}\delta-\frac{g_0}{n_0}\alpha_j-(-1)^j\frac{r\nu_{1j}}{n_0}$$

$$a_{3j}=1-\frac{g_0}{n_0}-\left(1+\frac{n_1+n_{12}}{n_0}\right)d_2+(-1)^j\frac{r\nu_{3j}}{n_0}$$

$$G_{ms}=\frac{1}{2\gamma}\left[-\gamma\delta_1-(-1)^m\frac{g_0}{n_0}\left(\frac{r}{R}\alpha_s+\delta\right)-(-1)^{m+s}\frac{r\nu_{2s}}{n_0}\right],\quad m,s=1,2$$

The solution of the equations obtained is determined using the scheme described in Section 3. As a result we obtain

$$F_1=\left[\frac{4g_0}{R(D_1^2+1)(D_3^2+1)}-\frac{2n_0R}{r^2}\right]H_4\delta$$

$$F_2=mR\omega^2+2\left(\frac{g_0r}{R^2\gamma^2}-\frac{n_0}{r}\right)\gamma\delta_1\,\mathrm{th}\,\pi\gamma+\frac{g_0}{R}\left(1-\frac{g_0r^2}{n_0R^2\gamma^2}\right)(\alpha_2-\alpha_1)$$

$$P=-r^{-1}n_0(\alpha_2-\alpha_1),\quad d_2=-\tfrac{1}{2}H_2^{-1}(\alpha_2-\alpha_1)$$

$$M_2=\left[\frac{2n_0R}{r}-\frac{6g_0r}{R(D_1^2+4)(D_3^2+4)}\right]H_4\delta$$

$$M_3=-\frac{4g_0rH_4\delta}{R(D_1^2+1)(D_3^2+1)},\quad \alpha_1+\alpha_2=-2(1+H_3)\frac{R}{r}\delta \tag{4.8}$$

The characteristics of the deformed tread $\alpha_2-\alpha_1$, $\alpha_1+\alpha_2$, δ, δ_1, d_2 and the moments M_2 and M_3 are found from (4.8) for arbitrary values of F_1, F_2 and P. We then find the relation between the angular velocities Ω and ω using the equation $\Omega=\omega(Rr^{-1}+\delta_1)$.

If we neglect dynamic effects in relations (3.11) and (4.8) between the forces, moments and quantities characterizing the deformation of the tread, by putting $g_0=0$, they agree with the corresponding relations obtained previously in [2, 3, 6].

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